Hilbert Series of Quasi-invariant Polynomials

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m-Quasiinvariance

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Definition

Let m be a non-negative integer and k be a field. A polynomial $F \in k[x_1, x_2, ..., x_n]$ is m-quasiinvariant if for all $1 \leq i < j \leq n$

$$(1 - s_{ij})F(x_1, x_2, ..., x_n)$$

is divisible by $(x_i - x_j)^{2m+1}$.

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$(k = \mathbb{C})$

- $F(x, y) = 2x^3 + 6xy^2 \in Q_1$ since $F(x, y) - F(y, x) = 2(x - y)^3$
- $F(x, y) = x^5 - 5x^3y^2 \in Q_1$ since $F(x, y) - F(y, x) = (x - y)^3(x^2 + 3xy + y^2)$

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($k = \mathbb{F}_2$)

- $F(x, y) = x^8 \in Q_3$ since $F(x, y) - F(y, x) = x^8 - y^8 = (x - y)^8$

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Definition

The Hilbert series of the space of m-quasiinvariants to be

$$HS_m(t) = \sum_{d \geq 0} t^d dim(Q_{m,d})$$

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- Q_m is a module over the ring of symmetric polynomials
- Closed under addition
- Closed under multiplication by ring elements(symmetric polynomials)
- Satisfies distributive property

More Properties

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- Q_m is a finitely generated module over the ring of symmetric polynomials
- Thus, $HS_m(t)$ can be written as

$$\frac{P(t)}{\prod_{i=1}^{n}(1 - t^i)}$$

where $P(t)$ is a polynomial with integer coefficients.

Previous Result

Theorem (Felder and Veselov)

Hilbert series of m-quasiinvariants in $\mathbb{C}$ is

$$HS_m(t) = n!t^{m\binom{n}{2}} \sum_{Youngdiagrams} \prod_{i=1}^{n} t^{m(l_i - a_i)+l_i} \frac{1-t^i}{h_i(1-t^{h_i})}$$

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- For example, when $n = 4$ and $m = 5$ the Hilbert series is $1 + t + 2t^2 + 3t^3 + 5t^4 + 6t^5 + 9t^6 + 11t^7 + 15t^8 \ldots$

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- For example, when $n = 4$ and $m = 5$ the Hilbert series is $1 + t + 2t^2 + 3t^3 + 5t^4 + 6t^5 + 9t^6 + 11t^7 + 15t^8 ...$
- Young diagrams are objects useful in representation theory
- Want to generalize in $\mathbb{F}_p$

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- Work with n=2
- Ex. If $g = x^2 + 5y^2$, the Hilbert series for the space of 2-quasiinvariants divisible by g is

$$\frac{t^5 + t^4}{(1-t)(1-t^2)}$$

Theorem

If $g = (ax^k + by^k)$ *and* $a^2 \neq b^2$
then the Hilbert series divisible by g *is*

$$t^k \left(\frac{t^{2m} + t^{2m+1} + \sum_{i=1}^{m} t^{2(m-i)+min(i,k)} - \sum_{i=1}^{m} t^{2(m-i)+min(i,k)+2}}{(1-t)(1-t^2)} \right)$$

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Theorem

If there exists integers $a \geq 1$, $k \geq 0$, and $b \geq 0$ such that

$$p^a(nk+1) + 2b\binom{n}{2} \leq mn$$

$$p^a(2k+1) + 2b \geq 2m+1,$$

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- If $a = 1, k = 0, b = 0$, then the Hilbert series is greater for $2m + 1 \leq p \leq mn$

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The previous conditions are necessary for the Hilbert series to be greater in $\mathbb{F}_p$ *than in* $\mathbb{C}$.

Conjecture

Furthermore, the minimal non-symmetric polynomial in $\mathbb{F}_p$ *is of the form*

$$G = P_k^{p^a} \prod_{1 \le i < j \le n} (x_i - x_j)^{2b}$$

where P_k *is a generator of degree* $kn + 1$ *in* $\mathbb{C}$.

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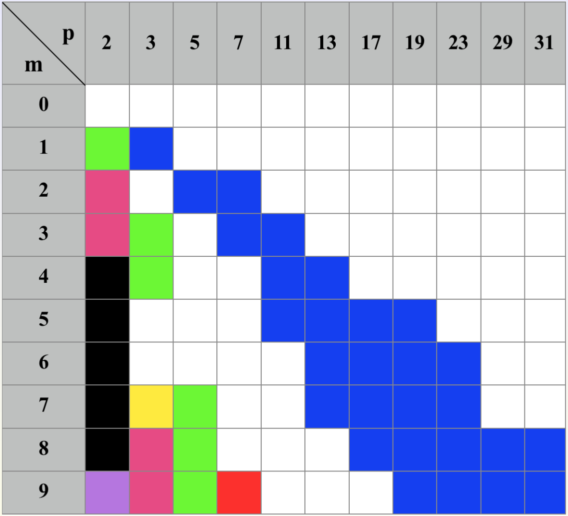

Figure: n=4

Future Studies

- Generalize results for first problem for generic g
- Compute Hilbert series for finite fields using the representation theory of the Cherednik algebra

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